Ref: Coutinho, A primer of algebraic D-modules
Nestruev, Smooth manifolds & observables

Differential operators (Sue, 28/2/2020)

$k = \mathbb{C}$

(I) Derivations

A = commutative $\mathbb{C}$-algebra (unital)

A **derivation** of A is a $\mathbb{C}$-linear map $\partial : A \to A$ s.t. $\partial(ab) = a\partial(b) + \partial(a)b$ $\forall a, b \in A$.

Exercise: $\partial(\mathbb{C}) = 0$

$\mathrm{Der}_{\mathbb{C}}(A) = \mathrm{Der}_{\mathbb{C}}(A, A)$

More generally, B commutative ring, A a B-algebra, M an A-bimodule we have $\mathrm{Der}_B(A, M) = \{ \partial \in \mathrm{Hom}_B(A, M) \mid \forall a, b \in A, \ \partial(ab) = a\partial(b) + \partial(a)b \}$

$\mathrm{Der}_{\mathbb{C}}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$ $\qquad A \hookrightarrow \mathrm{End}_{\mathbb{C}}(A)$

$a \longmapsto \mu_a : b \longmapsto ab$

Prop $\partial \in \mathrm{End}_{\mathbb{C}}(A)$ is a derivation $\Leftrightarrow$ $\partial(\mathbb{C}) = 0$ and $\forall a \in \underset{\mathrm{End}_{\mathbb{C}}(A)}{\overset{\cup}{A}}$ $\partial a - a\partial \in \underset{\mathrm{End}_{\mathbb{C}}(A)}{\overset{\cup}{A}}$

Proof

Claim Leibniz $\Leftrightarrow$ $\partial a - a\partial = \partial(a)$

Let $b \in A$ $\quad (\partial a - a\partial)(b) = \partial(ab) - a\partial(b) \overset{\text{Leibniz}}{=} \partial(a)b$

$(\Rightarrow)$ $\partial a - a\partial = \partial(a) \in A$

$(\Leftarrow)$ Let $\partial a - a\partial = c \in A$

$(\partial a - a\partial)(1) = c(1) = c$

$\quad \| $

$\partial(a) - 0 \qquad \therefore \ \partial(a) = \partial a - a\partial$.

Example $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x]) = \mathbb{C}[x] \cdot \frac{d}{dx}$

Why? $\supseteq$ ✓

$\subseteq$ Let $\partial \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$. Then $\partial = \partial(x)\frac{d}{dx}$.

$\left(\partial(x)\frac{d}{dx}\right)(x) = \partial(x)$ $\quad \therefore$ ∂ and $\partial(x)\frac{d}{dx}$ agree on $\mathbb{C}$ & x.

Leibniz & $\mathbb{C}$-linear $\Rightarrow$ they agree on $\mathbb{C}[x]$.

Likewise, $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \dots, x_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \dots, x_n] \frac{\partial}{\partial x_i}$

Example $A \in C^\infty(M)$ $\quad \mathrm{Der}_{\mathbb{R}}(A) = \mathfrak{X}(M)$

A still a commutative $\mathbb{C}$-algebra $(\ldots)$

(II) Differential operators

Def 1 The ring $D(A)$ of ($\mathbb{C}$-linear) differential operators on A is the subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A and $\mathrm{Der}_{\mathbb{C}}(A)$.

$\theta \in D(A)$ has order $(\leq)$ p if θ is a sum of products of $\leq p$ derivations.

eg: $\frac{d^2}{dx^2}+1 = \left(\frac{d}{dx}\right)^2+1$ has order 2.

Def 2 $D^0(A) := A$

$D^p(A) := \{\theta \in \mathrm{End}_{\mathbb{C}}(A) \mid \theta a - a\theta \in D^{p-1}(A) \ \forall a \in A\}$

differential operators of order $\leq p$.

$$D(A) = \bigcup D^p(A) \quad \text{since } D^{p+1}(A) \supseteq D^p(A)$$

Let $\theta \in D^1(A)$. Then $\theta = \underbrace{(\theta - \theta(1))}_{\in \mathrm{Der}_{\mathbb{C}}(A)} + \underbrace{\theta(1)}_{\in A}$ $\therefore$ $D^1(A) = \mathrm{Der}_{\mathbb{C}}(A) \oplus A$

$$D^p(A)\, D^r(A) \subseteq D^{p+r}(A)$$

Thm (Grothendieck) Def 1 = Def 2 $\Leftrightarrow$ $X = \mathrm{Spec}\,A$ is nonsingular

and if Def 1 = Def 2,

tensor algebra

A bimodule

$$D(A) = \frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\underbrace{\theta\otimes\theta' - \theta'\otimes\theta = [\theta,\theta']}_{\text{only for } \theta,\theta' \in \mathrm{Der}_{\mathbb{C}}(A)\,??}}$$

← as element of $\mathrm{End}_{\mathbb{C}}(A)$

eg: $A = \mathbb{C}[x]$

$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{C}[x]\frac{d}{dx} = W$ (Witt algebra)

$$D(A) = \frac{\mathbb{C}\langle x, \partial = \frac{d}{dx}\rangle}{\partial x - x\partial = 1} \quad \text{(Weyl algebra)}$$

Def 2 is the correct definition. Def 1 D's $\subseteq$ Def 2 D's. The question is whether one gets them all this way. eg: $D(\mathbb{C}[t^2, t^3])$ $\Big\{$ Def 1 $\neq$ Def 2.

Fact/Lemma $\theta \in D^p(A)$ $\theta' \in D^r(A)$ then $\theta \cdot \theta' - \theta' \cdot \theta \in D^{p+r-1}(A)$

$\therefore$ $D^1(A)$ is a LA, as is $\mathrm{Der}_{\mathbb{C}}(A)$.

From now on: all varieties are nonsingular: Spec A always nonsingular

Question X, Y affine varieties. If $D(X) \cong D(Y)$, is $X \cong Y$?
$X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$

False if X is allowed to be singular.
(eg: cuspidal rational curves)

(III) $D(A) \rightsquigarrow$ Poisson algebra

We saw that $[D^p(A), D^r(A)] \subseteq D^{p+r-1}(A)$

Check: if $\delta, \delta' \in \mathrm{Der}_{\mathbb{C}}(A)$, so is $\delta \circ \delta' - \delta' \circ \delta =: [\delta, \delta']$

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\
&= \delta\big(\delta'(a)b + a\delta'(b)\big) - \delta'\big(\delta(a)b + a\delta(b)\big) \\
&= \delta\delta'(a)b + \cancel{\delta'(a)\delta(b)} + \cancel{\delta(a)\delta'(b)} + a\delta\delta'(b) \\
&\quad - \delta'\delta(a)b - \cancel{\delta(a)\delta'(b)} - \cancel{\delta'(a)\delta(b)} - a\delta'\delta(b) \\
&= [\delta, \delta'](a)b + a[\delta, \delta'](b).
\end{aligned}$$

Def $\mathrm{gr}\, D(A) = \bigoplus_p D^p(A)/D^{p-1}(A)$

Prop $\mathrm{gr}\, D(A)$ is (1) a commutative ring and (2) a Poisson algebra

Proof Let $\pi \in D^p(A)$, $\rho \in D^r(A)$

(1) $\pi\rho, \rho\pi \in D^{p+r}(A)$ but $\pi\rho - \rho\pi \in D^{p+r-1}$

so $\underbrace{\pi\rho + D^{p+r-1}(A)}_{=: \mathrm{gr}(\pi\rho)} = \rho\pi + D^{p+r-1}(A)$ (& $\pi\rho$ well-defined).

(2) $\{\mathrm{gr}\,\pi, \mathrm{gr}\,\rho\} = \mathrm{gr}[\pi, \rho] = [\pi, \rho] + D^{p+r-2}(A)$

Poisson bracket b/c $[\,,\,]$ is a Lie bracket, ... ◩

In fact, $\mathrm{gr}\, D(A) = \mathrm{gr}\left(\dfrac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta = [\delta, \delta']}\right) \underset{\text{Thm!}}{=} \dfrac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta} = \mathrm{Sym}_A(\mathrm{Der}_{\mathbb{C}}(A))$

Let $X = \mathrm{Spec}\, A$, $\mathrm{Der}_{\mathbb{C}}(A) = \mathrm{Vect}(X) = \mathbb{C}[T^*X]$
polynomial functions

(IV) Weyl algebras $\quad A = \mathbb{C}[x_1,\dots,x_n]$

Facts about $D(A)$

$y_i = -\partial_i$

$$D(A) \cong \frac{\mathbb{C}\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle}{\substack{x\text{'s commute}\\ y\text{'s commute}\\ x_iy_j - y_jx_i = \delta_{ij}}}$$

n^{th} Weyl algebra

$\operatorname{gr} D(A) = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \quad \{x_i,y_j\} = \delta_{ij} \quad \{x_i,x_j\} = \{y_i,y_j\} = 0$

$T^*\mathbb{A}^n$

$D(A)$ is simple, $\operatorname{gr} D(A)$ is Poisson simple. (In noncomm. algebra simples are not fields!)

Prop.

Let I be a right ideal of $D(A)$. Then $J = \operatorname{gr}(I)$ is an ideal of $\operatorname{gr} D(A)$, and it's involutive: $\{J,J\} \subseteq J$
(coisotrope)

Proof Let $\theta,\eta \in I$. Then $\theta\eta - \eta\theta \in I$ (right ideal!). So $\{\operatorname{gr}\theta, \operatorname{gr}\eta\} = \operatorname{gr}([\theta,\eta]) \in J$

Thm. (Gabber) $\sqrt{J} := \{f \mid \exists n \text{ s.t. } f^n \in J\}$ is also a coisotrope.

Cor (Bernstein's inequality) $\dim(V(J) \subseteq \mathbb{C}^n) \geq n$
vanishing locus

Cor $D(\mathbb{C}[x])$ has no finite-dimensional modules.

Pf. Let V be a D-module with $\dim_{\mathbb{C}} V = d$.
D acts on $V \Rightarrow \exists\, X,Y \in \operatorname{Mat}_{d\times d}(\mathbb{C})$ s.t. $XY - YX = I \Rightarrow \operatorname{tr}(XY-YX) = d$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \, \overset{\shortparallel}{0} \Rightarrow\!\Leftarrow$. ■

Example Let $A = \mathbb{C}[x,y]$ with $\{x,y\} = 1$.
Let $J = \langle x^2, xy, y^2\rangle$ with $\sqrt{J} = \langle x,y\rangle$.
Here J is a coisotrope, but $\sqrt{J}$ is not.
Presumably $J \neq \operatorname{gr}(I)$ for any right ideal I of $D(A)$?